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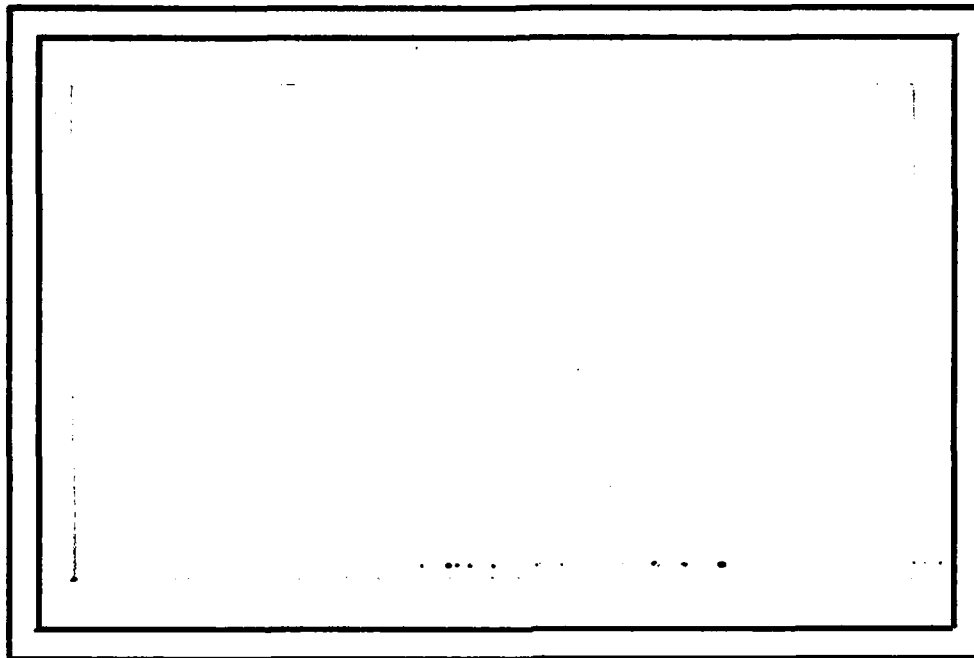ELECTE
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THREE-DIMENSIONAL
DIGITAL TOPOLOGY

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ABSTRACT

Basic concepts of connectedness, cavities, and holes
are defined for subsets of three-dimensional arrays.
Three-dimensional arcs and curves are also defined and
characterized.

1. Introduction

Geometrical properties of subsets of digital pictures play an important role in computer image analysis and recognition [1]. In particular, there is a well-developed theory [2] of "topological" properties such as connectedness for subsets of two-dimensional arrays.

The analysis of three-dimensional arrays has become of increasing interest with the rapid growth of computed tomography, in which discrete 3D representations of solid objects are reconstructed from sets of projections. 3D arrays can also be obtained from sets of cross-sections in microscopy; and time sequences of images can also be regarded as 3D arrays in which the third dimension is time. Thus it has become desirable to study the geometrical properties of subsets of 3D arrays.

Some early work on 3D digital geometry was done by Gray [3; see also 4], and several theoretical papers on digital topology also considered generalizations to higher dimensions [5,6]; but the basic 3D theory has not yet been systematically presented. This paper is part of a planned series on 3D digital geometry; a report on 3D digital convexity has already been issued [7], and a paper on the theory of 3D digital surfaces is in preparation. (It will deal with surfaces as "thin" objects, as opposed to [8], in which surfaces are composed of the "cracks" between objects.) The present paper deals with basic concepts of 3D connectedness and with 3D digital arcs and curves.

2. Connectedness and distance

Let Σ be a 3D array of lattice points, which we may assume without loss of generality to be $n \times n \times n$, e.g. $\Sigma = \{(i,j,k) \mid 1 \leq i,j,k \leq n\}$. Let S be a nonempty subset of Σ; we can regard S as specified by a mapping from Σ into $\{0,1\}$, where the points of S are those that map into 1, so that we can refer to points of S as 1's, and to points of the complement $\overline{S}$ of S as 0's. The points of Σ are sometimes [8] called "voxels" (short for "volume elements"; analogous to "pixels" = "picture elements" in two dimensions).

Any $(i,j,k) \in \Sigma$ has three types of <u>neighbors</u> (some of which may not exist if (i,j,k) is on the border of Σ):

a) Six "face neighbors": $(i{\pm}1,j,k), (i,j{\pm}1,k)$, and $(i,j,k{\pm}1)$

b) Twelve "edge neighbors": $(i{\pm}1,j{\pm}1,k)$, $(i,j{\pm}1,k{\pm}1)$, and $(i{\pm}1,j,k{\pm}1)$, where the two signs in each triple are chosen independently

c) Eight "corner neighbors": $(i{\pm}1,j{\pm}1,k{\pm}1)$, where all three signs are chosen independently.

This nomenclature corresponds to regarding (i,j,k) as the center of a unit cube; then the face (edge, corner) neighbors of (i,j,k) are the centers of the unit cubes that share a face (edge, corner) with (i,j,k)'s cube.

We will call the face neighbors "6-neighbors", and all three kinds of neighbors "26-neighbors", and we will consider only these two types of neighbors. If A,B are disjoint subsets of Σ, we say that A and B are 6-adjacent if some point of A is a 6-neighbor of some point of B; "26-adjacent" is defined analogously.

A path π is a sequence $P_0, P_1, \ldots, P_m$ of points (e.g., $P_h = (i_h, j_h, k_h)$) such that P_i is a neighbor of P_{i-1}, $1 \leq i \leq m$. Note that this is two definitions in one, depending on whether "neighbor" means "6-neighbor" or "26-neighbor"; π can be a "6-path" or a "26-path".

Two points P,Q are said to be connected in S if there exists a path $P = P_0, P_1, \ldots, P_m = Q$ from P to Q consisting entirely of points of S. Evidently, "connected" is an equivalence relation (P is connected to P for any $P \in S$; if P is connected to Q, then Q is connected to P; if P is connected to Q and Q to R, then P is connected to R). This relation partitions S into equivalence classes (=maximal sets of points each pair of which is connected in S). These classes are called the connected components of S. Here again we have two definitions, and can speak of 6- or 26-connectedness and of 6- or 26-components.

An algorithm for labelling the 6-components of a given set S is presented in [4]; it is analogous to the standard two-dimensional algorithm, and makes use of a plane-by-plane, row-by-row scan of Σ. The details are straightforward, and the 26-case is also analogous.

The 26-neighbors of a given point P can be defined relative to P by specifying, for each coordinate, whether it is incremented by 1, unchanged, or decremented by 1. Thus each neighbor is specified by a 3-digit ternary number $\alpha\beta\gamma$, where $\alpha,\beta,\gamma \in 0,1,2$ (or we can use -1 or 1' instead of 2, for brevity); here 000 represents P itself, and the 6-neighbors are those triples in which exactly one of α,β and γ is 1 or 1'. (Compare [9], which suggests using 5-bit binary numbers, somewhat inefficiently, to represent the 26-neighbors. Note that in the 2D case, it is efficient to use 3-bit binary numbers for the 8 neighbors of a point, but one could also use 2-digit ternary numbers to represent the neighbors as well as the point itself.)

The <u>city block distance</u> between two points (x,y,z) and (u,v,w) is defined as $|x-u|+|y-v|+|z-w|$, and the <u>chessboard distance</u> between them is defined as $\max[|x-u|,|y-v|,|z-w|]$. These are exactly analogous to the 2D definitions, and can be immediately extended to digital arrays of any number of dimensions. Readily, they are metrics on Σ.

Just as in the 2D case, it is easily shown that the city block (chessboard) distance between two points is the length of a shortest 6-path (26-path) between the points. In particular, the points whose city block (chessboard) distance from P is 1 are just the 6-neighbors (26-neighbors) of P.

We can also define distance within a given connected set S in terms of paths that lie in S, where the type of path (6- or 26-) corresponds to the type of connectedness used for S.

Specifically, for any P,Q in S, we define the intrinsic distance $d_s(P,Q)$ as the length of a shortest path in S from P to Q. Readily, this too is a metric.

As an application of the concept of intrinsic distance, we can prove, exactly as in the 2D case [2], that any connected set S contains points whose deletion does not disconnect S. (The proof given in [2] is incorrectly stated in terms of ordinary, rather than intrinsic, distance; we give the correct version here.)

Proof: Let P be any point of S, and let $Q \in S$ be such that $d_s(P,Q)$ is a local maximum, i.e., $d_s(P,Q') \le d_s(P,Q)$ for all neighbors Q' of Q. We show that every point of S-{Q} is connected to P, so that S-{Q} is connected. Let π be a path from P to (say) R in S, and let Q' be the point just after the last occurrence of Q on π (if there are no occurrences, we are done). Let π' be a shortest path in S from P to Q'; then Q cannot occur on π', since if it did, π' would be strictly longer than $d_s(P,Q) \ge d_s(P,Q')$. Hence π', together with the part of π from Q' to R, is a path from P to R in S-{Q}.

3. Cavities, surroundedness, and borders

We assume from now on that S does not meet the border of
Σ, i.e., that for all $(i,j,k) \in S$ we have $1 < i,j,k < n$), so that
the border of Σ consists entirely of 0's. Thus if we now de-
fine connectedness and components for the complement $\overline{S}$ of S,
exactly one of these components contains the border of Σ.
This component will be called the background of S; all other
components of $\overline{S}$ (if any) will be called cavities in S.

In the 2D case, non-background components of $\overline{S}$ are called
holes. In 3D, we can also define "holes" (e.g., a ring has
a hole), but their definition is not so simple; it will be dis-
cussed in Sections 4 and 5.

It turns out to be desirable to use opposite types of con-
nectedness for S and for $\overline{S}$--i.e., if we use the 6-definitions
for S, then we use the 26-definitions for $\overline{S}$, and vice versa.
(Analogously, in 2D, if we use 4-neighbor definitions for S,
then we use 8-neighbor definitions for $\overline{S}$.) This convention
assures that various concepts to be introduced later, such as
borders and genus, are well-behaved.

Let A,B be subsets of Σ. We say that A surrounds B, or B
is surrounded by A, if any path from (a point of) B to (a point
of) the border of Σ must meet (i.e., contain a point of) A.
More generally, let A,B,C be subsets of Σ; we say that B
separates A from C if any path from A to C must meet B. Thus
A surrounds B iff. A separates B from the border of Σ.

Proposition 1. Any S surrounds its cavities, and is surrounded
by its background.

Proof: Since the cavities and background are in different com-
ponents of $\bar{S}$, a path from a cavity to the border (which is a
subset of the background) cannot consist entirely of points of
$\bar{S}$, so must meet S. On the other hand, a path from S to the
border must meet the background, since the border is a subset
of the background. Note that "surrounds" in the first part of
this proposition must be understood in the sense of $\bar{S}$'s connected-
ness, i.e., if we use the 6- (26-) definitions for $\bar{S}$, then we
must use 6- (26-) paths in defining "surrounds".

Let C be a component of S and D a component of $\bar{S}$. The
remainder of this section deals with adjacency and surrounded-
ness relations between such components. Note first that in
considering adjacency between components of S and components of
$\bar{S}$, it does not matter whether we use 6- or 26-adjacency, by
virtue of

Proposition 2. If a component C of S and a component D of $\bar{S}$
are 26-adjacent, they are also 6-adjacent.

Proof: Suppose, for example, that C is 6-connected, D is 26-
connected, and that $(0,0,0) \in C$ and $(1,1,1) \in D$. If $(1,0,0)$ is
in $\bar{S}$, it's in D (since it's 26-adjacent to $(1,1,1)$), and we're
done. If not, $(1,0,0)$ is in S, hence in C, and $(1,1,0)$ is
either in S, hence in C, or in $\bar{S}$, hence in D, making C and D
6-adjacent in either case. The proofs in other cases are
analogous.

Let C be a component of S, and D a component of $\bar{S}$ that is adjacent to C. The set of points of C that are adjacent to points of D (in the sense of the connectedness of D) is called the D-border of C. The C-border of D is defined analogously.

Proposition 3. The D-border of C is connected (in the C sense).

In 2D, this is proved by defining a border following algorithm,proving that it visits the entire D-border of C, and observing that the points it visits are all connected to the starting point. The situation in 3D is more complicated, since the border points cannot be visited in a simple sequence. . . . For a proof that borders are connected in three (or more) dimensions, see [8].

Once we know that borders are connected, we have the following results just as in the 2D case:

Corollary 4. Let D_1 and D_2 be distinct components of $\bar{S}$ that are adjacent to C; then C separates D_1 from D_2.

Proof: If D_1 and D_2 were in the same component D of $\bar{C}$, they would both meet the D-border of C, which is impossible since they are different components of $\bar{S}$, and the D-border is a connected subset of $\bar{S}$.

Corollary 5. The adjacency graph of S (i.e., the graph whose nodes are the components of S and $\bar{S}$, and where two nodes are joined by an arc iff the corresponding components are adjacent) is a tree.

Proof: Clearly this graph is connected, and by Corollary 4 it can have no cycles.

<u>Corollary 6</u>. Let C,D be adjacent components of $S,\bar{S}$, respectively; then either C surrounds D or vice versa. Moreover, exactly one component of $\bar{S}$ surrounds any given component of S (and vice versa, for nonbackground component of $\bar{S}$).

<u>Proof</u>: By Corollary 4, two D's cannot be in the same component of $\bar{C}$; hence at most one can be in the background component, so that all others are surrounded by C. The D_0 that contains (e.g.) the point to the right of a rightmost point of C cannot be surrounded by C; and any path from C to the border of Σ, when it last leaves C, must enter a component of $\bar{S}$ that is not surrounded by C, hence must enter D_0, so that D_0 surrounds C.

<u>Corollary 7</u>. The adjacency tree of S can be regarded as a directed tree, rooted at the background component of $\bar{S}$, under the relationship of surroundedness.

4. Holes: the two-dimensional case

As pointed out at the beginning of Section 3, some 3D
objects have holes (in the sense that a ring has a hole);
these holes are not the same as cavities, which we defined
as non-background connected components of the object's comple-
ment. In the next section we will give a definition for the
class of objects that have (no) holes, in terms of the (non)-
existence of a closed curve in the object that is not equiva-
lent--in a sense to be defined below--to a degenerate closed
curve consisting of a single point. This definition is analo-
gous to the one used in ordinary topology. Before giving the
3D definition, we give an analogous 2D definition and prove
that it does in fact characterize 2D objects that have (no)
holes. We assume familiarity with the material in [2]. In
particular, a **simple closed curve** γ is a connected set of
points each of which has exactly two neighbors in the set, so
that the points can be arranged in a cyclic sequence, and each
is adjacent to just its predecessor and successor in the sequence.
This definition allows some degenerate cases, e.g., $\gamma = 1, 11$, or $\begin{smallmatrix} 11 \\ 11 \end{smallmatrix}$,
if γ is a 4-curve.

Let $\gamma = P_1, \ldots, P_m$ and $\gamma' = P_1', \ldots, P_n'$ be simple closed curves
(possibly degenerate) that are contained in a given set S. We
say that γ and γ' are **strongly equivalent** in S if there exists
a point P_i of γ and a run of consecutive points $P_h', \ldots, P_k'$ of γ'
(or vice versa) such that

a) $P_{i+1}, \ldots, P_m, P_1, \ldots, P_{i-1} = P'_{k+1}, \ldots, P'_n, P'_1, \ldots, P'_{h-1}$—i.e., γ with P_i deleted is the same as γ' with $P'_h, \ldots, P'_k$ deleted

b) $P'_h, \ldots, P'_k$ are all adjacent to P_i

Here "adjacent" is to be understood in the sense opposite to that in which γ and γ' are curves, i.e., if they are 4-curves, it means 8-adjacent, and vice versa. The reflexive, transitive closure of strong equivalence in S is called equivalence in S; in other words, γ and γ' are called equivalent in S if $\gamma = \gamma'$, or if there exists a sequence of curves $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_r = \gamma'$, all subsets of S, such that γ_i is strongly equivalent to γ_{i-1}, $1 \le i \le r$. We assume in the following proposition that γ and γ' are 4-curves; the argument in the other case is similar.

Note first that P_i in the definition of strong equivalence has exactly two 4-neighbors, P_{i-1} and P_{i+1}, that lie on γ; and it has some 8-neighbors that lie inside γ (if γ is non-degenerate) and some that lie outside. Now P_{i-1} and P_{i+1} break up the cyclic sequence of 8-neighbors of P_i into two non-null runs (P_{i-1} and P_{i+1} cannot be 4-adjacent), and since these runs cannot cross γ, one of them lies (on or) inside γ and the other outside. Moreover, one of these runs must be $P'_h, \ldots, P'_k$, since these points are 8-neighbors of P_i, do not occur in γ, and join P_{i-1} to P_{i+1}.

Proposition 8. If γ and γ' are strongly equivalent, any point inside γ is on or inside γ', and vice versa.

Proof: This is not hard to see if γ or γ' is degenerate; suppose both are nondegenerate. Let P be inside γ but outside γ', so that there is an 8-path π from P to the picture border that does not meet γ', but does meet γ. Thus π must meet γ at P_i, and we can assume (shortening π if necessary) that it contains P_i only once, so that the point preceding P_i on π is inside γ and the point following P_i is outside γ. But these points are 8-neighbors of P_i, hence belong to the two runs of neighbors mentioned in the previous paragraph, so that one of them belongs to the 4-arc $P'_h, \ldots, P'_k$ of γ', contradiction.

Conversely, let P be inside γ' but outside γ. This means there is an 8-path π' from P to the picture border that meets γ' but not γ, so that π' meets the 4-arc $P'_h, \ldots, P'_k$, say at P'_j. If this arc consists of the 8-neighbors of P_i that lie inside γ, π' cannot get from P'_j to the picture border without crossing γ, contradiction. Thus we may assume that the arc consists of the 8-neighbors of P_i that lie outside γ.

To conclude the proof, we first observe that in the case we are now considering, P_i lies inside γ'. Indeed, suppose we had an 8-path ρ from P_i to the picture border (and not returning to P_i) that did not meet γ'; thus ρ leaves P_i via a neighbor that lies inside γ. But ρ cannot get from such a neighbor to the picture border without crossing γ, and whether it crosses at P_i or at another point (which thus lies on γ'), we have a contradiction. Now let P'_j be the first point at which π'

meets γ'. As before, P_j' has two 4-neighbors P_{j-1}', P_{j+1}' that

lie on γ', and they divide its remaining 8-neighbors into two

runs, one consisting of points inside γ', the other of points

outside γ'. Since P_i is a neighbor of P_j' and lies inside γ',

it belongs to the first run. Since $P_{i-1}, P_{i+1}, P_h', \ldots, P_k'$ are all

8-neighbors of P_i and are all on γ', it is not hard to see

that this run must in fact consist solely of P_i. But the point

on π' just preceding P_j' is inside γ', and the only 8-neighbor

of P_j' that lies inside P_j' is P_i, so that π' does meet γ, con-

tradiction.

<u>Corollary 9</u>. Let γ and γ' be equivalent curves in S; then every

point of $\overline{S}$ inside γ is also inside γ' (and vice versa).

<u>Proof</u>: If γ and γ' are strongly equivalent in S, this follows

from Proposition 8, since a point of $\overline{S}$ cannot be on γ' (or γ).

The corollary now follows by induction from the definition of

equivalence.

<u>Proposition 10</u>. Let C be a simply connected component of S,

and let β be the (outer) border of C, so that $C-\beta$ is the interior

of C. Let $C_1, \ldots, C_k$ be the components of $C-\beta$ (in the $\overline{S}$ sense), and for

$1 \le i \le k$, let β_i be the set of points of β that are adjacent to C_i

(in the $\overline{S}$ sense). Then each β_i is a curve in the S sense.

<u>Proof</u>: If we start at a point of C_i and move along any path

(in the sense of C_i's connectedness), the point at which we leave

C_i cannot be in another C_j, since C_i and C_j are different

components, and it cannot be outside C, since we cannot leave
C without crossing its border; hence this point must be in β,
and thus in β_i, so that β_i surrounds C_i. Since C_i is connected,
and disjoint from β_i, it is a component of the complement of β_i,
and is not the background component. If $\overline{\beta}_i$ had another compo-
nent, it would be adjacent to β_i, hence adjacent to β. But
since β is the border of C, the only points adjacent to points
of β are in C, $\overline{C}$, or β. A point of C adjacent to β_i is in C_i
by definition; and a point of $\beta - \beta_i$ adjacent to β_i is in the
background component of $\overline{\beta}_i$, since it is adjacent to $\overline{C}$, while
a point of $\overline{C}$ adjacent to β_i is certainly in the background com-
ponent of $\overline{\beta}_i$. Thus $\overline{\beta}_i$ has exactly two components, and every
point of it is adjacent to both components (to C_i by definition
of β_i, and to the background component since every point of
$\beta_i \subseteq \beta$ is adjacent to $\overline{C}$). By the converse to the Jordan Curve
Theorem [2], this proves that β_i is a curve.

Note that if an outer border is a curve, its interior is
connected, by the Jordan Curve Theorem; but if the interior
of an outer border is connected, the border may not be a curve;
even though there is only one C_1, still β_1 may not be all of β
$$\text{xxx}$$
(example: C is xxxxx).
$$\text{xxx}$$

Corollary 11. If C has a hole, there is a simple closed curve
in C that surrounds the hole.

Proof: A hole H cannot meet the border β of C. Temporarily
fill all holes in C; the interior of the resulting C^* is
nonempty, since H is in it. By Proposition 10, any
connected component C_i of the interior of C^*

has a border $\beta_i \subseteq \beta$ that is a simple closed curve. Since holes cannot meet β, these β_is are subsets of C, and H must be a subset of one of the C_is (H cannot meet two of them, since it is connected and cannot meet the border of either). Thus β_i surrounds H.

A curve in C will be called <u>reducible</u> in C if it is equivalent in C to a degenerate curve consisting of a single point of C.

<u>Corollary 12</u>. If C has a hole, it contains a nonreducible curve.

<u>Proof</u>: By Corollary 11, it contains a curve that surrounds the hole; by Corollary 9, any equivalent curve must also surround the hole.

<u>Proposition 13</u>. If C has no holes (i.e., is simply connected), every curve in C is reducible.

<u>Proof</u>: The inside of any curve γ is surrounded by γ, hence by C, and since C surrounds no 0's, the inside of γ consists entirely of 1's. Now readily the inside of any curve γ is simply connected. By [2], a simply connected set of 1's that has two or more points has ends (points with just one neighbor 1), e.g. $\underline{1}$ 1) or corners (e.g. $\frac{1}{1}\,\frac{1}{1}$). Since the other neighbors are not 1's, but are adjacent to the 1's, they must be points of γ, e.g.

```
c c c     c c c
c 1 1 or  c 1 1 . Thus at any end or corner we can construct a
c c c     c 1 1
```

strongly equivalent curve γ', using the underlined 1, e.g.

```
c c       c c
c' 1 or   c c' 1 . The inside of this γ' has fewer points than
c c       c 1 1
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that of γ, and is still simply connected. Repeating this

argument, we can reduce the inside of γ to a single point, e.g.

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c c c
c 1 c
c c c
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, and this curve is strongly equivalent to the one-point

curve c', so that the original γ is reducible.

Corollary 12 and Proposition 13 give us

Theorem 14. C has no holes iff every curve in C is reducible.

In the next section, we will use the 3D analog of Theorem

14 as a definition of sets that have no holes, in terms of

every curve being reducible. Analogously, one might define 3D

sets that have no cavities in terms of the reducibility of

simple closed surfaces, and prove that this is equivalent to

the definition in terms of components of the complement; but we

will not define simple closed surfaces in this paper.

5. Arcs, curves, holes, and genus

A simple arc is a path that does not cross or touch itself, i.e. P_i is a neighbor of P_j iff $|i-j|=1$ (from which it follows that $P_i \neq P_j$ unless $i=j$). [Here and in what follows, we have two definitions, depending on whether we use the prefix 6- or 26- for the terms "arc", "curve", "path", "neighbor", "connected", etc.] Equivalently, we can define an arc as a connected set α of points each of which has exactly two neighbors in α, with two exceptions, called the endpoints, that have only one neighbor each. The proof that these definitions are equivalent is exactly as in the 2D case [2]. (Note that in the second definition, we must also allow a single point to be an arc.)

A simple closed curve is an arc whose endpoints are adjacent, i.e. a path such that P_i is a neighbor of P_j iff $|i-j|=1$ (modulo n+1), where n is the path length. Equivalently, we can define a curve as a connected set γ of points each of which has exactly two neighbors in γ. (The proof of equivalence is again analogous to that in [2].) Here there are a number of degenerate cases that satisfy the definition, e.g. $\gamma = \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$ in the 6-connected case, or 1 or 11 in either case.

Proposition 16. No curve is both a 6-curve and a 26-curve. An arc is both a 6-arc and a 26-arc iff it is a straight line segment parallel to one of the coordinate axes.

Proof: If α is both 6- and 26-, the neighbors of any $P \in \alpha$ must be opposite 6-neighbors, since otherwise they would be

26-neighbors of each other; hence α can only extend in one principal direction. In particular, there can be no such γ, since its points could only get farther apart, so it could never close.

Using this definition of curve, we can now define (strong) equivalence of curves, and reducible curves, just as we did for the 2D case in Section 4. We can then define a set C to have no holes iff every curve in C is reducible.

Proposition 17. An arc has no holes.

Proof: It is easily seen that an arc α can only contain degenerate curves, which are all reducible.

Proposition 18. A nondegenerate curve has a hole.

Proof: It is easily seen that the only nondegenerate curve contained in a curve γ is γ itself, and that γ is not reducible in γ.

Proposition 19. An arc or curve has no cavities.

Proof: It is not hard to see that for any P,Q in $\bar{\alpha}$ (or $\bar{\gamma}$), and any path π from P to Q, we can divert π to go around α, if necessary, so that $\bar{\alpha}$ has only one component.

In 2D, the genus of a set S is the number of its components minus the number of its holes (or, equivalently, the number of components of S, minus the number of components of $\bar{S}$, plus 1). The genus can be computed by counting various types of local patterns in Σ [1,2,3]; in fact, it can be shown [10] that the only topological properties of S that can be computed in this way are functions of the genus.

Extensions of these ideas to 3D were introduced in [3].
In this case the genus can be regarded as the number of compo-
nents minus the number of holes plus the number of cavities.
Here again, it can be computed by counting various types of
local patterns; for example, if we use 6-connectedness for S
and 26-connectedness for $\overline{S}$, we have

<u>Proposition 20</u>. The genus is equal to $n_1-n_2+n_4-n_8$, where

n_1 is the number of 1's

n_2 is the number of 1x1x2 blocks of 1's (in all
orientations)

n_4 is the number of 1x2x2 blocks of 1's (in all
orientations)

n_8 is the number of 2x2x2 blocks of 1's

For a detailed discussion of this result see [4]. Since the
genus is not a very useful property, further details on its
computation will not be given here; see [3]. Note, however,
that if we apply this formula to a single connected set C all
of whose cavities have been filled, we can use it as a basis
for defining the number of holes in C.

6. Concluding remarks

This paper has introduced some of the basic concepts of digital topology for 3D arrays, involving connectedness, cavities, holes, arcs, and curves. The much more difficult task of defining digital surfaces will be treated in a subsequent paper. As in the 2D case, these concepts are needed in order to properly define various algorithms for processing 3D arrays. We need to understand connectedness in order to define algorithms for counting objects; and we need to understand arcs and surfaces in order to define connectedness-preserving thinning algorithms, since the result of thinning a rod-like object should be a set of arcs, and the result of thinning a plate-like object should be a set of surfaces. 3D thinning algorithms will also be the subject of a forthcoming report.

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Basic concepts of connectedness, cavities, and holes are defined
for subsets of three-dimensional arrays. Three-dimensional
arcs and curves are also defined and characterized.

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